C. Woodward - 30/5/08 - Seidel's exact triangle for fibered Dehn twists

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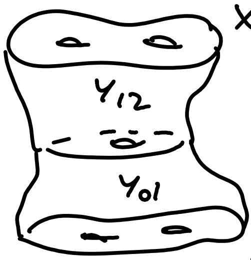
Last time:

Decorated cobordisms ⟶ generalized Lagr. correspondences ⟶ functors

(Decorated cobordisms ⟶ functors: category-valued TFT)

- obj = connected surfaces
  w/ principal $U(r)$-bundle P
  and connection δ on $\det(P)$
  $d = \deg P$, $r = \mathrm{rk}$ coprime
- morphisms =

$\underline{Y} =$ 

$$\longmapsto \quad \underline{L}(\underline{Y}) = \begin{bmatrix} L(Y_{12}) \\ L(Y_{01}) \end{bmatrix} \quad \longmapsto \quad \text{quilt functor}$$

corresp^s between moduli of connections with central curvature $\sim \delta$

$$M(X_-) \longrightarrow M(X_+)$$

$$\Phi(\underline{L}(\underline{Y})) : \quad \mathrm{Don}^{\#}(M(X_-)) \longrightarrow \mathrm{Don}^{\#}(M(X_+))$$

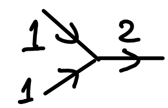
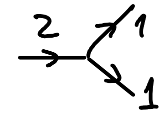
Theory with graphs:

Def:

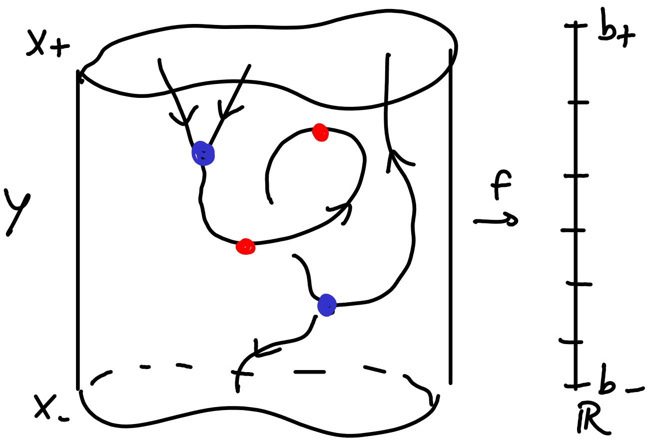
- A <u>marked surface</u> :=
  - surface X
  - $\underline{x} = (x_1 \dots x_n)$ distinct oriented pts in X
  - $\underline{\mu} : \underline{x} \longrightarrow \{1, 2\}$ ← for simplicity (in g^al : <u>weights</u>)
- $M(X, \underline{x}, \underline{\mu}) := \left\{ \begin{array}{l} \text{flat } SU(r) \text{ bundles} / X - \underline{x} \text{ with holonomy} \\ \text{around } x_j := \begin{cases} \exp(\mathrm{diag}(1, 1, \dots, 1, 1-r)\frac{2\pi i}{r})^{\pm 1} & \text{if } \mu_j = 1 \\ \exp(\mathrm{diag}(2, \dots, 2, 2-r, 2-r)\frac{2\pi i}{r})^{\pm 1} & \text{if } \mu_j = 2 \end{cases} \end{array} \right\} \Big/ SU(r)$

smooth, monotone, compact, symplectic if r coprime to $\sum \mu_j$

- Let $(X_\pm, \underline{x}_\pm, \underline{\mu}_\pm)$ marked surfaces: a <u>cobordism w/ graph</u> from $(X_-, \underline{x}_-, \underline{\mu}_-)$ to $(X_+, \underline{x}_+, \underline{\mu}_+)$:= $\left\{ \begin{array}{l} Y \text{ cobordism } X_- \to X_+ \\ \Gamma \text{ oriented graph} \subset Y, \quad \Gamma \cap X_\pm = \underline{x}_\pm \\ \underline{\mu} : \mathrm{edges}(\Gamma) \to \{1, 2\} \end{array} \right.$

NB: μ must be compatible with $\mu_\pm$ at boundaries, and its behavior at vertices is only  or 

- Given such (Y, Γ, μ):



$f : Y \to \mathbb{R}$ Morse function,
$f|_{edges(\Gamma)}$ morse
f injective on $crit(f) \sqcup crit(f|_\Gamma) \sqcup Vert(\Gamma)$
$b_- = b_0 < b_1 < \ldots < b_m = b_+$ separating regular levels cut (Y, Γ) into elementary cobordisms

$\Rightarrow$ $\underline{L}(Y,\Gamma) = L(Y_{01}, \Gamma_{01}) \# \ldots \# L(Y_{m-1,m}, \Gamma_{m-1,m})$
gen^d Lagr. correspondence $M(X_-, \underline{x}_-, \underline{x}_+) \to M(X_+, \underline{x}_+, \mu_+)$

Lemma: If $Y = X \times [b_-, b_+]$, then for a generic perturbation of height function this is well-defined and $\underline{L}(Y,\Gamma)$ is independent of the choice of perturbation.

Minimal Maslov number $= 2 \Rightarrow \partial^2_{CF(\underline{L}_0, \underline{L}_1)} \neq 0$ in general (but central)
(for case $SU(2)$ it's ok; for $SU(r)$ not clear).

$(CF(\underline{L}_0, \underline{L}_1), \partial)$ is now a matrix factorization
& independent of choices up to chain homotopy

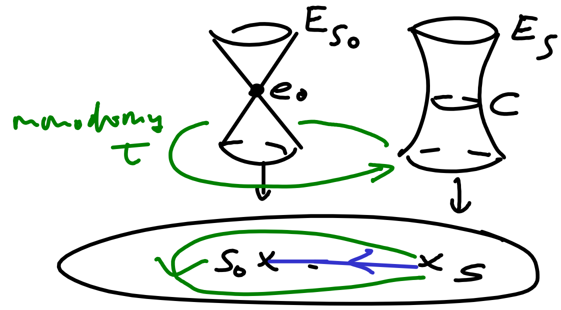
$\rightsquigarrow$ $Don^\#(M(X, \underline{x}, \mu))$ category enriched in matrix factorizations

Result: TFT with values in enriched categories.

§ Exact triangles for Dehn twists (after Seidel)

$\pi : E \to S$ Lefschetz fibration,

$\mathrm{crit}(\pi) = e_0 \in E_{s_0}$



Picard-Lefschetz: $\tau^* : H^*(E_s) \circlearrowleft$ is given by $\tau^* = \mathrm{Id} + (-1)^{(n+1)(n+2)/2} (C, \cdot) C$

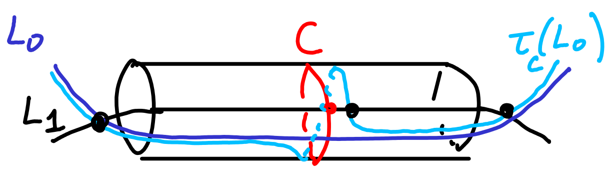
where C = vanishing cycle $\begin{cases} n \text{ even: reflection} \\ n \text{ odd: shear} \end{cases}$

Ingredients: (1) C is a sphere (Lagrangian in sympl. case)

(2) τ is a generalized Dehn twist about C

(geodesic flow by π on C, 0 away from C)

(3) Count intersection points



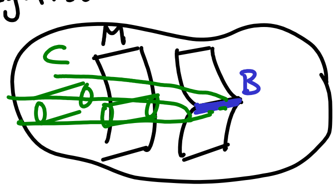
$\rightsquigarrow (\tau_C(L_0), L_1) = (L_0, L_1) \pm (L_0, C)(L_1, C)$

- More general singularities (Pham, Clemens, Landsman)

Def: A Lefschetz-Bott fibration allows π to have Morse-Bott sings.

$M \overset{i}{\hookleftarrow} C \overset{p}{\longrightarrow} B = \mathrm{crit}(\pi)$ vanishing bundle

reg. fiber



$\overline{C} = (p \times i)(C) \subset B \times M$

Monodromy: $\tau^* = id + (-1)^{(c+1)(c+2)/2} [\overline{C}]^t . [\overline{C}] . -$

dual $H^*(B) \to H^*(M)$

correspondence $H^*(M) \to H^*(B)$

(in Morse case: $H^*(M) \to H^*(pt)$ $[C] \cap$.)

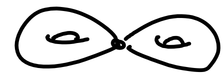
Let $E \xrightarrow{\pi} S$ symplectic Lefschetz-Bott fibration ($\rightsquigarrow so(c+1) \dots$)

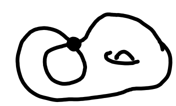
<u>Perutz</u> $\rightsquigarrow$ τ is a fibered Dehn twist around C,
C = spherically fibered coisotropic $\subset M$

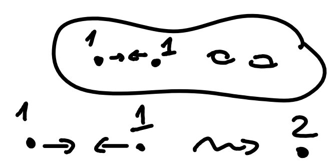
<u>Seidel</u> $\rightsquigarrow$ given such a twist, $\exists$ Lefschetz-Bott fibration with this monodromy.

<u>Examples:</u> (1) $E \to S$ Lefschetz fibration of curves w/ data,
$(r,d) = (2,1)$ (degree 1 rank 2 bundles)

$\Rightarrow M(E) = \bigcup_{s \neq s_0} M(E_s)$ has a completion to a Lefschetz-Bott fibration
rel. moduli

vanishing cycles = codim. 1 for E_{s_0} = 

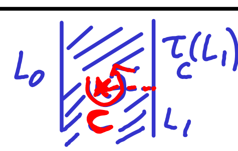
codim. 3 for E_{s_0} = 

- Similarly, Lefschetz fibration of marked surfaces
  $\rightsquigarrow$ codim. 2 v.c. for E_{s_0} : where 

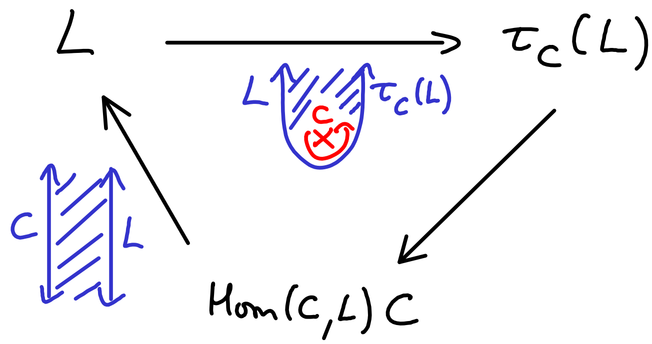
- Also get Morse-Bott fibrations when considering e.g. relative Hilbert scheme of a LF, ...

<u>Seidel's long exact sequence:</u> under suitable monotonicity conditions,

$$HF(L_0, L_1) \xrightarrow{\text{count}} HF(L_0, \tau_C(L_1)) \to HF(L_0, C) \otimes HF(C, L_1) \xrightarrow{\text{count}} HF(L_0, L_1)$$



In fact, <u>Fukaya-version</u> (cf. Seidel's book):

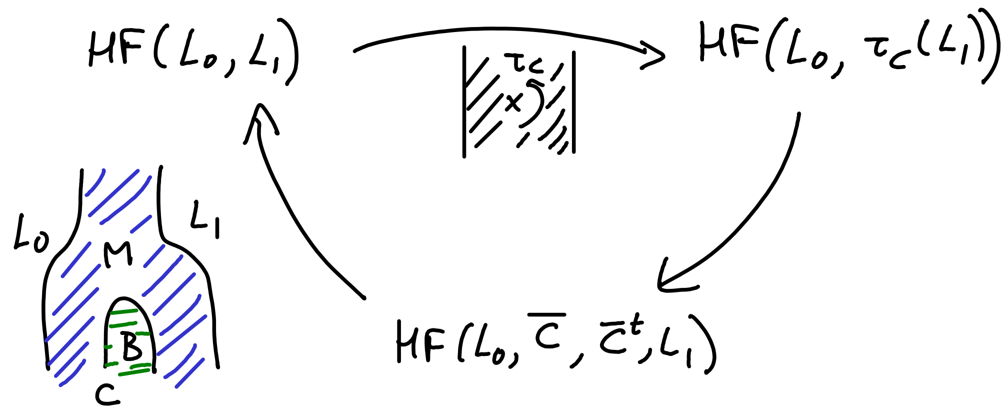


exact triangle in Fukaya cat.

For fibered Dehn twists: Quilted version:

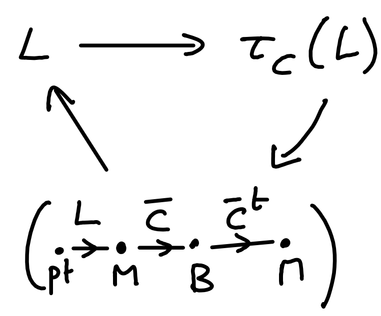
$\overline{C} \subset M^- \times B$ Lagr. correspondence for a Morse-Bott singularity

Thm: Under suitable monotonicity hypotheses,



· Work in progress := Fukaya category version

in $DFuk^{\#}(M)$:



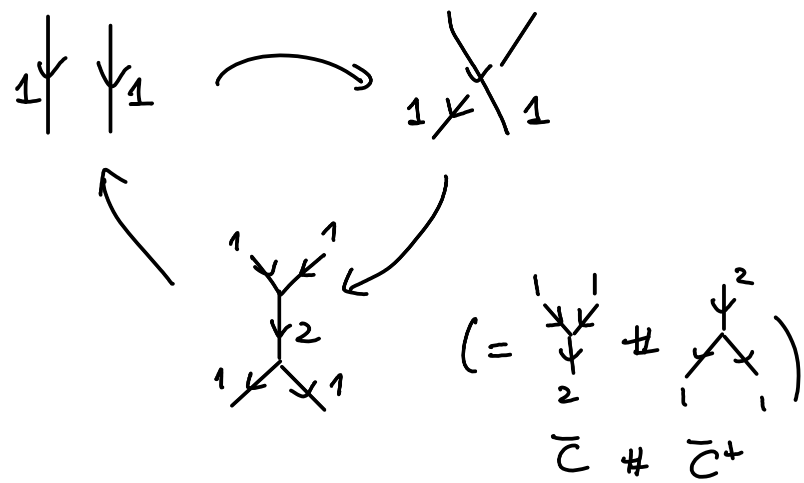
generalized brane

More generally, in $DFuk^{\#}(M\times M) \hookrightarrow Fun(DFuk^{\#}(M), DFuk^{\#}(M))$

expect

$$\begin{array}{ccc} \Delta & \longrightarrow & graph(\tau_C) \\ & \nwarrow \quad \swarrow & \\ & M \xrightarrow{\overline{C}} B \xrightarrow{\overline{C}^t} M & \end{array}$$

- Applying this to graph TFT: (& suitable fibered twists in it)



as in Khovanov-Rozansky!!